

On cosmological-type solutions in multi-dimensional model with Gauss-Bonnet term

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Abstract

A $(n + 1)$ -dimensional Einstein-Gauss-Bonnet (EGB) model is considered. For diagonal cosmological-type metrics, the equations of motion are reduced to a set of Lagrange equations. The effective Lagrangian contains two “minisuperspace” metrics on \mathbb{R}^n . The first one is the well-known 2-metric of pseudo-Euclidean signature and the second one is the Finslerian 4-metric that is proportional to n -dimensional Berwald-Moor 4-metric. When a “synchronous-like” time gauge is considered the equations of motion are reduced to an autonomous system of first-order differential equations. For the case of the “pure” Gauss-Bonnet model, two exact solutions with power-law and exponential dependence of scale factors (with respect to “synchronous-like” variable) are obtained. (In the cosmological case the power-law solution was considered earlier in papers of N. Deruelle, A. Toporensky, P. Tretyakov and S. Pavluchenko.) A generalization of the effective Lagrangian to the Lowelock case is conjectured. This hypothesis implies existence of exact solutions with power-law and exponential dependence of scale factors for the “pure” Lowelock model of m -th order.

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1 Introduction

Here we deal with D -dimensional gravitational model with the Gauss-Bonnet term. The action reads

$$S = \int_M d^D z \sqrt{|g|} \{ \alpha_1 R[g] + \alpha_2 \mathcal{L}_2[g] \}, \quad (1.1)$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric defined on the manifold M , $\dim M = D$, $|g| = |\det(g_{MN})|$ and

$$\mathcal{L}_2 = R_{MNPQ} R^{MNPQ} - 4R_{MN} R^{MN} + R^2 \quad (1.2)$$

is the standard Gauss-Bonnet term. Here α_1 and α_2 are constants. The appearance of the renormalizable Gauss-Bonnet term as well as quadratic Riemann curvature terms in multidimensional gravity is motivated by string theory [1, 2, 3, 4, 5]. (For a review of fourth-order gravity in $D = 4$, see [6].)

At present, the so-called Einstein-Gauss-Bonnet (EGB) gravity and its modifications are intensively used in cosmology, see [7, 8] (for $D = 4$), [9, 11, 12, 13, 14, 15, 16] and references therein, e.g. for explanation of accelerating expansion of the Universe following from supernovae (type Ia) observational data [17]. Certain exact solutions in multidimensional EGB cosmology were obtained in [9]-[16] and some other papers.

EGB gravity is also intensively investigated in a context of black-hole physics. The most important results here are related with the well-known Boulware-Deser-Wheeler solution (corresponding to the Schwarzschild-Tangherlini solution in general relativity) [18, 19] and its generalizations [20, 21, 22, 23], for a review and references, see [24, 25]. For certain applications of brane-world models with Gauss-Bonnet term, see review [26] and references therein.

Here we are interested in the cosmological (type) solutions with diagonal metrics (of Bianchi-I-like type) governed by scale factors depending upon one variable.

For $\alpha_2 = 0$, we have the Kasner type solution with the metric

$$g = -d\tau \otimes d\tau + \sum_{i=1}^n A_i^2 \tau^{2p^i} dy^i \otimes dy^i, \quad (1.3)$$

where $A_i > 0$ are arbitrary constants, $D = n + 1$ and parameters p^i obey the relations

$$\sum_{i=1}^n p^i = 1, \quad (1.4)$$

$$\sum_{i=1}^n (p^i)^2 = 1 \quad (1.5)$$

and hence

$$\sum_{1 \leq i < j \leq n} p^i p^j = \frac{1}{2} \left(\sum_{i=1}^n p^i \right)^2 - \frac{1}{2} \sum_{i=1}^n (p^i)^2 = 0. \quad (1.6)$$

For $D = 4$, this is the well-known Kasner solution [27]. The set of eqs. (1.4), (1.5) is equivalent to the set of eqs. (1.4), (1.6).

In [10], a Einstein-Gauss-Bonnet (EGB) cosmological model was considered. For “pure” Gauss-Bonnet (GB) case $\alpha_1 = 0$ and $\alpha_2 \neq 0$, N. Deruelle has obtained a cosmological solution with the metric (1.3) for $n = 4, 5$ and parameters obeying the relations

$$\sum_{i=1}^n p^i = 3, \quad (1.7)$$

$$\sum_{1 \leq i < j < k < l \leq n} p^i p^j p^k p^l = 0. \quad (1.8)$$

It was reported by A. Toporensky and P. Tretyakov in [13] that this solution was verified by them for $n = 6, 7$. In recent paper by S. Pavluchenko [28] the power-law solution was verified for all n (and also generalized to the Lowelock case [34]).

In this paper we give a derivation of the “power-law” (cosmological type) solution for arbitrary n . We also show that for $D \neq 4$ this solution in “pure” GB cosmology is unique in a class of solutions with power-law dependence of scale factors: $a_i(\tau) = A_i \tau^{p^i}$, when the parameters p^1, \dots, p^n contain more than two non-zero numbers. When $(n - 2)$ parameters among p^i are zero, say $p^3 = \dots = p^n = 0$, then the metric (1.3) obeys the equations of motion (for $\alpha_1 = 0$) for arbitrary values of two Kasner-like parameters (say p^1, p^2).

The numerical analysis of cosmological solutions in EGB gravity for $D = 5, 6$ [15] shows that the singular “power-law” solutions (1.3), (1.7), (1.8) (e.g. with a little generalization of scale factors $a_i(\tau) = A_i(\tau_0 \pm \tau)^{p^i}$, where τ_0 is constant) appear as asymptotical solutions for certain initial values as well as Kasner-type solutions (1.3)-(1.5) do.

The paper is organized as follows. In Section 2 the equations of motion for $(n + 1)$ -dimensional EGB model are considered. For diagonal cosmological type metrics the equations of motion are reduced to a set of Lagrange equations corresponding to certain “effective” Lagrangian (in agreement with [10, 28] for cosmological case). Section 3 is devoted to the case of the “pure” Gauss-Bonnet model. Two exact solutions: with power-law and exponential dependence of scale factors (with respect to “synchronous-like” variable) are obtained. In Section 4 the equations of motion are reduced to an autonomous system of first order differential equations (when a “synchronous-like” time gauge is considered). For $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ it is shown that for any non-trivial solution with exponential dependence of scale factors $a_i(\tau) = A_i \exp(v^i \tau)$, $i = 1, \dots, n$, there are no more than three different numbers among v^1, \dots, v^n . In Section 5 a generalization of the effective Lagrangian to the Lowelock case is conjectured and exact solutions with power-law and exponential dependence of scale factors for the “pure” Lowelock model of m -th order are presented. (See also [10, 28] for “power law” cosmological solutions.) Certain useful relations and proofs are collected in Appendix.

2 The cosmological type model and its effective Lagrangian

2.1 The set-up

Here we consider the manifold

$$M = \mathbb{R}_* \times M_1 \times \dots \times M_n, \quad (2.1)$$

with the metric

$$g = we^{2\gamma(u)} du \otimes du + \sum_{i=1}^n e^{2\beta^i(u)} \varepsilon_i dy^i \otimes dy^i, \quad (2.2)$$

where $w = \pm 1$ and any M_i is 1-dimensional manifold with the metric $g^i = \varepsilon_i dy^i \otimes dy^i$, $\varepsilon_i = \pm 1$, $i = 1, \dots, n$. Here and in what follows $\mathbb{R}_* = (u_-, u_+)$ is an open subset in \mathbb{R} . (Here we identify g^i with $\hat{g}^i = p_i^* g^i$ which is the pullback of the metric g^i to the manifold M by the canonical projection: $p_i : M \rightarrow M_i$, $i = 1, \dots, n$.) The functions $\gamma(u)$ and $\beta^i(u)$, $i = 1, \dots, n$, are smooth on $\mathbb{R}_* = (u_-, u_+)$.

For $w = -1$, $\varepsilon_1 = \dots = \varepsilon_n = 1$ the metric (2.2) is a cosmological one while for $w = 1$, $\varepsilon_1 = -1$, $\varepsilon_2 = \dots = \varepsilon_n = 1$ it describes static configurations.

According to Appendix A, the integrand in (1.1), when the metric (2.2) is substituted, reads as follows

$$\sqrt{|g|} \{ \alpha_1 R[g] + \alpha_2 \mathcal{L}_2[g] \} = L + \frac{df}{du}, \quad (2.3)$$

where

$$L = \alpha_1 L_1 + \alpha_2 L_2, \quad (2.4)$$

$$L_1 = (-w) e^{-\gamma + \gamma_0} G_{ij} \dot{\beta}^i \dot{\beta}^j, \quad (2.5)$$

$$L_2 = -\frac{1}{3} e^{-3\gamma + \gamma_0} G_{ijkl} \dot{\beta}^i \dot{\beta}^j \dot{\beta}^k \dot{\beta}^l, \quad (2.6)$$

$\gamma_0 = \sum_{i=1}^n \beta^i$ and

$$G_{ij} = \delta_{ij} - 1, \quad (2.7)$$

$$G_{ijkl} = (\delta_{ij} - 1)(\delta_{ik} - 1)(\delta_{il} - 1)(\delta_{jk} - 1)(\delta_{jl} - 1)(\delta_{kl} - 1) \quad (2.8)$$

are respectively the components of two “minisuperspace” metrics on \mathbb{R}^n . (For cosmological case see also [10, 28, 37].) The first one is the well-known 2-metric of pseudo-Euclidean signature: $\langle v_1, v_2 \rangle = G_{ij} v_1^i v_2^j$ and the second one is the Finslerian 4-metric: $\langle v_1, v_2, v_3, v_4 \rangle = G_{ijkl} v_1^i v_2^j v_3^k v_4^l$, $v_s = (v_s^i) \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot, \cdot, \cdot \rangle$ are respectively 2- and 4-linear symmetric forms on \mathbb{R}^n . (Here we denote $\dot{A} = dA/du$ etc.)

In (2.3) the function $f = f(\gamma, \beta, \dot{\beta})$ has the following form:

$$f = \alpha_1 f_1 + \alpha_2 f_2, \quad (2.9)$$

where f_1 and f_2 are defined in Appendix A (see (A.22) and (A.23)).

The derivation of (2.4)-(2.6) is based on the relations obtained in Appendix A (see (A.20), (A.21)) and the following identities

$$G_{ij}v^iv^j = \sum_{i=1}^n (v^i)^2 - \left(\sum_{i=1}^n v^i\right)^2, \quad (2.10)$$

$$\begin{aligned} G_{ijkl}v^iv^jv^kv^l &= \left(\sum_{i=1}^n v^i\right)^4 - 6\left(\sum_{i=1}^n v^i\right)^2 \sum_{j=1}^n (v^j)^2 \\ &+ 3\left(\sum_{i=1}^n (v^i)^2\right)^2 + 8\left(\sum_{i=1}^n v^i\right) \sum_{j=1}^n (v^j)^3 - 6\sum_{i=1}^n (v^i)^4. \end{aligned} \quad (2.11)$$

The first identity (2.10) is a trivial one. The second one (2.11) may be verified by straightforward calculations (see Appendix B).

It follows immediately from the definitions (2.7) and (2.8) that

$$G_{ij}v^iv^j = -2 \sum_{i<j} v^iv^j, \quad (2.12)$$

$$G_{ijkl}v^iv^jv^kv^l = 24 \sum_{i<j<k<l} v^iv^jv^kv^l. \quad (2.13)$$

Due to (2.13), $G_{ijkl}v^iv^jv^kv^l$ is zero for $n = 1, 2, 3$ ($D = 2, 3, 4$). For $n = 4$ ($D = 5$), $G_{ijkl}v^iv^jv^kv^l = 24v^1v^2v^3v^4$ and our 4-metric is proportional to the well-known Berwald-Moor 4-metric [29, 30] (see also [31, 32] and references therein). We remind the reader that the 4-dimensional Berwald-Moor 4-metric obeys the relation: $\langle v, v, v, v \rangle_{BM} = v^1v^2v^3v^4$. The Finslerian 4-metric with components (2.8) coincides up to a factor with the n -dimensional analogue of the Berwald-Moor 4-metric.

2.2 The equations of motion

The equations of motion corresponding to the action (1.1) have the following form

$$\mathcal{E}_{MN} = \alpha_1 \mathcal{E}_{MN}^{(1)} + \alpha_2 \mathcal{E}_{MN}^{(2)} = 0, \quad (2.14)$$

where

$$\mathcal{E}_{MN}^{(1)} = R_{MN} - \frac{1}{2}Rg_{MN}, \quad (2.15)$$

$$\begin{aligned} \mathcal{E}_{MN}^{(2)} &= 2(R_{MPQS}R_N^{PQS} - 2R_{MP}R_N^P \\ &- 2R_{MPNQ}R^{PQ} + RR_{MN}) - \frac{1}{2}\mathcal{L}_2g_{MN}. \end{aligned} \quad (2.16)$$

The field equations (2.14) for the metric (2.2) are equivalent to the Lagrange equations corresponding to the Lagrangian L from (2.4). This follows from the relations

$$\mathcal{E}_{00}(-2w) \exp(\gamma_0 - \gamma) = \frac{\partial L}{\partial \gamma}, \quad (2.17)$$

$$\mathcal{E}_{ii}(-2\varepsilon_i) \exp(\gamma + \gamma_0 - 2\beta^i) = \frac{\partial L}{\partial \beta^i} - \frac{d}{du} \frac{\partial L}{\partial \dot{\beta}^i}, \quad (2.18)$$

$$\mathcal{E}_{0i} = 0, \quad (2.19)$$

$i = 1, \dots, n$.

Formulas (2.17)-(2.19) may be verified just by straightforward calculations based on the relations for the Riemann tensor from Appendix A. But there exists a more “economic” way to prove these formulas using: **(i)** the diagonality of the matrix \mathcal{E}_{MN} (in coordinates $(y^M) = (y^0 = u, y^i)$); **(ii)** the dependence of this matrix only on one variable u , i.e. $\mathcal{E}_{MN} = \mathcal{E}_{MN}(u)$; **(iii)** the relation (2.3). The proof of (2.17)-(2.19) is given in Appendix C.

Thus, equations (2.14) read as follows

$$w\alpha_1 G_{ij} \dot{\beta}^i \dot{\beta}^j + \alpha_2 e^{-2\gamma} G_{ijkl} \dot{\beta}^i \dot{\beta}^j \dot{\beta}^k \dot{\beta}^l = 0, \quad (2.20)$$

$$\begin{aligned} & \frac{d}{du} [-2w\alpha_1 G_{ij} e^{-\gamma+\gamma_0} \dot{\beta}^j] \\ & - \frac{4}{3} \alpha_2 e^{-3\gamma+\gamma_0} G_{ijkl} \dot{\beta}^j \dot{\beta}^k \dot{\beta}^l] - L = 0, \end{aligned} \quad (2.21)$$

$i = 1, \dots, n$. Due to (2.20)

$$L = -w \frac{2}{3} e^{-\gamma+\gamma_0} \alpha_1 G_{ij} \dot{\beta}^i \dot{\beta}^j. \quad (2.22)$$

3 Exact solutions in Gauss-Bonnet model

Now we put $\alpha_1 = 0$ and $\alpha_2 \neq 0$, i.e. we consider the cosmological type model governed by the action

$$S_2 = \alpha_2 \int_M d^D z \sqrt{|g|} \mathcal{L}_2[g]. \quad (3.1)$$

The equations of motion (2.14) in this case read

$$\mathcal{E}_{MN}^{(2)} = \mathcal{R}_{MN}^{(2)} - \frac{1}{2} \mathcal{L}_2 g_{MN} = 0, \quad (3.2)$$

where

$$\begin{aligned} \mathcal{R}_{MN}^{(2)} = & 2(R_{MPQS} R_N^{PQS} - 2R_{MP} R_N^P \\ & - 2R_{MPNQ} R^{PQ} + R R_{MN}). \end{aligned} \quad (3.3)$$

Due to identity $g^{MN} \mathcal{R}_{MN}^{(2)} = 2\mathcal{L}_2$, the set of eqs. (3.2) for $D \neq 4$ implies

$$\mathcal{L}_2 = 0. \quad (3.4)$$

It is obvious that the set of eqs. (3.2) is equivalent for $D \neq 4$ to the following set of equations

$$\mathcal{R}_{MN}^{(2)} = 0. \quad (3.5)$$

Equations of motion (2.20) and (2.21) in this case read as follows

$$G_{ijkl}\dot{\beta}^i\dot{\beta}^j\dot{\beta}^k\dot{\beta}^l = 0, \quad (3.6)$$

$$\frac{d}{du} \left[e^{-3\gamma+\gamma_0} G_{ijkl}\dot{\beta}^i\dot{\beta}^j\dot{\beta}^k\dot{\beta}^l \right] = 0, \quad (3.7)$$

$i = 1, \dots, n$. Here $L = 0$ due to (3.6).

Let us put $\ddot{\beta}^i = 0$ for all i or, equivalently,

$$\beta^i = c^i u + c_0^i, \quad (3.8)$$

where c^i and c_0^i are constants, $i = 1, \dots, n$. We also put

$$3\gamma = \gamma_0 = \sum_{i=1}^n \beta^i, \quad (3.9)$$

i.e. a modified “harmonic” variable is used. Recall that in the case $\alpha_1 \neq 0$ and $\alpha_2 = 0$, the choice $\gamma = \gamma_0$ corresponds to the harmonic variable u [33].

Then eqs. (3.7) are satisfied identically and eq. (3.6) gives us the following constraint

$$G_{ijkl}c^i c^j c^k c^l = 24 \sum_{i<j<k<l} c^i c^j c^k c^l = 0. \quad (3.10)$$

Thus, we have obtained a class of exact cosmological type solutions for the Gauss-Bonnet model (3.1) that is given by the metric (2.2) with the functions $\beta^i(u)$ and $\gamma(u)$ from (3.8) and (3.9), respectively, and integration constants c^i obeying (3.10).

3.1 Solution with power-law dependence of scale factors

Let us consider the solutions with

$$\sum_{i=1}^n c^i \neq 0. \quad (3.11)$$

Introducing the synchronous-type variable

$$\tau = \frac{1}{c} \exp(cu + c_0), \quad (3.12)$$

where

$$c = \frac{1}{3} \sum_{i=1}^n c^i, \quad c_0 = \frac{1}{3} \sum_{i=1}^n c_0^i, \quad (3.13)$$

and defining new parameters

$$p^i = c^i/c, \quad A_i = \exp[c_0^i + p^i(\ln c - c_0)], \quad (3.14)$$

$i = 1, \dots, n$, we get the “power-law” solution with the metric

$$g = wd\tau \otimes d\tau + \sum_{i=1}^n \varepsilon_i A_i^2 \tau^{2p^i} dy^i \otimes dy^i, \quad (3.15)$$

where $w = \pm 1$, $\varepsilon_i = \pm 1$; $A_i > 0$ are arbitrary constants, and parameters p^i obey the relations

$$\sum_{i=1}^n p^i = 3, \quad (3.16)$$

$$G_{ijkl} p^i p^j p^k p^l = 24 \sum_{i < j < k < l} p^i p^j p^k p^l = 0, \quad (3.17)$$

following from (3.10), (3.11) and (3.14). This solution is a singular one for any set of parameters p^i , see Appendix D.

In the cosmological case when $w = -1$, $\varepsilon_i = 1$ (for all i), this solution was obtained earlier in [10] for $D = 5, 6$ and verified recently in [28] for all $D > 4$.

Example 1. Let us consider the case $D = 6$ and $p_i \neq 0$, $i = 1, \dots, 5$. Relations (3.16) and (3.17) read in this case as follows

$$p^1 + p^2 + p^3 + p^4 + p^5 = 3, \quad (3.18)$$

$$p^1 p^2 p^3 p^4 p^5 \left(\frac{1}{p^1} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \frac{1}{p^5} \right) = 0. \quad (3.19)$$

Let us put $p^1 = x > 0$, $p^2 = \frac{1}{x}$, $p^3 = z > 0$, $p^4 = y < 0$, $p^5 = \frac{1}{y}$. Then we get

$$x + \frac{1}{x} + z + y + \frac{1}{y} = 3, \quad x + \frac{1}{x} + \frac{1}{z} + y + \frac{1}{y} = 0. \quad (3.20)$$

Subtracting the second relation in (3.20) from the first one we obtain $z - \frac{1}{z} = 3$ or $z = \frac{1}{2}(3 + \sqrt{13})$ ($z > 0$). For any $x > 0$ there are two solutions $y = y_{\pm}(x) = \frac{1}{2}(-A \pm \sqrt{A^2 - 4})$, where $A = x + \frac{1}{x} + \frac{1}{z} > 2$.

Proposition 1. *For $D \neq 4$ the metric (3.15) is a solution to equations of motion (3.2) if and only if the set of parameters $p = (p^1, \dots, p^n)$ either obeys the relations (3.16) and (3.17), or $p = (a, b, 0, \dots, 0)$, $(a, 0, b, 0, \dots, 0), \dots$, where a and b are arbitrary real numbers.*

This proposition is proved in Appendix E. (For cosmological solutions in dimensions $D = 5, 6$ see also [10].)

For $D = 4$ the metric (3.15) gives a solution to equations of motion (3.2) for any set of parameters p^i .

3.2 Solution with exponential dependence of scale factors

Now we consider the solution with

$$\sum_{i=1}^n c^i = 0. \quad (3.21)$$

Introducing the synchronous-type variable

$$\tau = u \exp(c_0), \quad (3.22)$$

where c_0 is defined in (3.13) and defining new parameters

$$v^i = c^i \exp(-c_0), \quad B_i = \exp(c_0^i), \quad (3.23)$$

$i = 1, \dots, n$, we are led to the cosmological-type solution with the metric

$$g = w d\tau \otimes d\tau + \sum_{i=1}^n \varepsilon_i B_i^2 e^{2v^i \tau} dy^i \otimes dy^i, \quad (3.24)$$

where $w = \pm 1$, $\varepsilon_i = \pm 1$; $B_i > 0$ are arbitrary constants, and parameters v^i obey the relations

$$\sum_{i=1}^n v^i = 0, \quad (3.25)$$

$$G_{ijkl} v^i v^j v^k v^l = 24 \sum_{i < j < k < l} v^i v^j v^k v^l = 0, \quad (3.26)$$

following from (3.10), (3.21) and (3.23).

Example 2. Let $D = 6$ and $v_i \neq 0$, $i = 1, \dots, 5$. Relations (3.25) and (3.26) read in this case as follows

$$v^1 + v^2 + v^3 + v^4 + v^5 = 0, \quad (3.27)$$

$$v^1 v^2 v^3 v^4 v^5 \left(\frac{1}{v^1} + \frac{1}{v^2} + \frac{1}{v^3} + \frac{1}{v^4} + \frac{1}{v^5} \right) = 0. \quad (3.28)$$

We put $v^1 = x > 0$, $v^2 = \frac{1}{x}$, $v^3 = 1$, $v^4 = y < 0$, $v^5 = \frac{1}{y}$. Then we get

$$x + \frac{1}{x} + 1 + y + \frac{1}{y} = 0, \quad (3.29)$$

For any $x > 0$ there are two solutions $y = y_{\pm}(x) = \frac{1}{2}(-B \pm \sqrt{B^2 - 4})$, where $B = x + \frac{1}{x} + 1 \geq 3$.

3.3 Some other solutions

The solutions to equations of motion (3.6) and (3.7) are not exhausted by relations (3.8)-(3.10). We give an example of another solution for $D > 4$:

$$e^{-3\gamma + \beta^1 + \beta^2 + \beta^3} \dot{\beta}^1 \dot{\beta}^2 \dot{\beta}^3 = C, \quad (3.30)$$

$$\beta^i(u) = \beta_0^i, \quad i > 3, \quad (3.31)$$

where β_0^i ($i > 3$) and C are arbitrary constants. In terms of “synchronous” variable τ (obeying $d\tau = e^{\gamma(u)} du$) this solutions reads as follows

$$g = w d\tau \otimes d\tau + \sum_{i=1}^n \varepsilon_i a_i^2(\tau) dy^i \otimes dy^i, \quad (3.32)$$

where

$$\left(\frac{da_1}{d\tau}\right) \left(\frac{da_2}{d\tau}\right) \left(\frac{da_3}{d\tau}\right) = C, \quad (3.33)$$

$$a_i(\tau) = a_i^0, \quad i > 3, \quad (3.34)$$

where $a_0^i > 0$ ($i > 3$) and C are constants. This solution contains a special solution with

$$a_1(\tau) = a_2(\tau) = a_3(\tau) = A\tau. \quad (3.35)$$

For $C = 0$ we get a special solution with arbitrary (smooth) functions $\gamma(u)$, $\beta^1(u)$, $\beta^2(u)$ and constant $\beta^i(u) = \beta_0^i$, for $i > 2$. In terms of synchronous variable this solution is described by the metric (3.32) with

$$a_1(\tau), a_2(\tau) - \text{arbitrary}, \quad a_i(\tau) = a_i^0 - \text{constant}, \quad i > 2. \quad (3.36)$$

Remark 1. For $D = 4$, or $n = 3$, the equations of motion (3.6) and (3.7) are satisfied identically for arbitrary (smooth) functions $\beta^i(u)$ and $\gamma(u)$. This is in agreement with that fact that in dimension $D = 4$, the action (3.1) is a topological invariant and its variation is identically zero.

4 Reduction to an autonomous system of first order differential equations

Now we put $\gamma = 0$, i.e. “the synchronous-like” time gauge is considered. We denote $u = \tau$. By introducing “Hubble-like” variables $h^i = \dot{\beta}^i$, we rewrite eqs. (2.20) and (2.21) in the following form

$$w\alpha_1 G_{ij} h^i h^j + \alpha_2 G_{ijkl} h^i h^j h^k h^l = 0, \quad (4.1)$$

$$\begin{aligned} & \left[-2w\alpha_1 G_{ij} h^j - \frac{4}{3}\alpha_2 G_{ijkl} h^j h^k h^l \right] \sum_{i=1}^n h^i \\ & + \frac{d}{d\tau} \left[-2w\alpha_1 G_{ij} h^j - \frac{4}{3}\alpha_2 G_{ijkl} h^j h^k h^l \right] - L = 0, \end{aligned} \quad (4.2)$$

$i = 1, \dots, n$, where

$$L = -w\alpha_1 G_{ij} h^i h^j - \frac{1}{3}\alpha_2 G_{ijkl} h^i h^j h^k h^l. \quad (4.3)$$

Due to (4.1),

$$L = -\frac{2}{3}w\alpha_1 G_{ij}h^i h^j. \quad (4.4)$$

Thus, we are led to the autonomous system of the first-order differential equations on $h^1(\tau), \dots, h^n(\tau)$.

Here we may use the relations (2.10), (2.11) and the following formulas (with $v^i = h^i$)

$$G_{ij}v^j = v^i - S_1, \quad (4.5)$$

$$G_{ijkl}v^j v^k v^l = S_1^3 + 2S_3 - 3S_1 S_2 + 3(S_2 - S_1^2)v^i + 6S_1(v^i)^2 - 6(v^i)^3, \quad (4.6)$$

$i = 1, \dots, n$, where $S_k = S_k(v) = \sum_{i=1}^n (v^i)^k$. Relation (4.6) is derived in Appendix B.

Let us consider the fixed point of the system (4.1) and (4.2): $h^i(\tau) = v^i$ with constant v^i corresponding to the solutions

$$\beta^i = v^i \tau + \beta_0^i, \quad (4.7)$$

where β_0^i are constants, $i = 1, \dots, n$. In this case we obtain the metric (3.24) with exponential dependence of scale factors. For $\alpha_1 = 0$ we get the solution (3.24)-(3.26).

Now we put $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. For the fixed point $v = (v^i)$ we have the set polynomial equations

$$G_{ij}v^i v^j - \alpha_w G_{ijkl}v^i v^j v^k v^l = 0, \quad (4.8)$$

$$\left[2G_{ij}v^j - \frac{4}{3}\alpha_w G_{ijkl}v^j v^k v^l \right] \sum_{i=1}^n v^i - \frac{2}{3}G_{ij}v^i v^j = 0, \quad (4.9)$$

$i = 1, \dots, n$, where $\alpha_w = \alpha_2(-w)/\alpha_1$. For $n > 3$ this is a set of forth-order polynomial equations.

The trivial solution $v = (v^i) = (0, \dots, 0)$ corresponds to a flat metric g .

For any non-trivial solution v we have $\sum_{i=1}^n v^i \neq 0$ (otherwise one gets from (4.9) $G_{ij}v^i v^j = \sum_{i=1}^n (v^i)^2 - (\sum_{i=1}^n v^i)^2 = 0$ and hence $v = (0, \dots, 0)$).

Let us consider the isotropic case $v^1 = \dots = v^n = a$. The set of equations (4.8) and (4.9) is reduced to the equation

$$n(n-1)a^2 + \alpha_w n(n-1)(n-2)(n-3)a^4 = 0. \quad (4.10)$$

For $n = 1$, a is arbitrary and $a = 0$ for $n = 2, 3$. When $n > 3$, the non-zero solution to eq. (4.10) exists only if $\alpha_w < 0$ and in this case

$$a = \pm \frac{1}{\sqrt{|\alpha_w|(n-2)(n-3)}}. \quad (4.11)$$

In cosmological case $w = -1$, this solution takes place when $\alpha_2/\alpha_1 < 0$.

Here the problem of classification of all solutions to eqs. (4.8), (4.9) for given n arises. Some special solutions of the form $(a, \dots, a, b, \dots, b)$, e.g. in a context of cosmology with

two factor spaces, for certain dimensions were considered in literature. See, for example, [9, 11, 12, 16].

Here we outline three properties of the solutions to the set of polynomial equations (4.8), (4.9).

Proposition 2. *For any solution $v = (v^1, \dots, v^n)$ to polynomial eqs. (4.8) and (4.9):*

- i) the vector $-v = (-v^1, \dots, -v^n)$ is also a solution;*
- ii) for any permutation σ of the set of indices $\{1, \dots, n\}$ the vector $v = (v^{\sigma(1)}, \dots, v^{\sigma(n)})$ is also a solution;*
- iii) there are no more than three different numbers among v^1, \dots, v^n , when $v = (v^1, \dots, v^n) \neq (0, \dots, 0)$.*

Proof. The first item of the proposition is trivial. The second one follows just from relations (2.10), (2.11), (4.5) and (4.6).

Now we prove the item **iii**). Let us suppose that there exists a non-trivial solution $v = (v^1, \dots, v^n)$ with more than three different numbers among v^1, \dots, v^n . Due to (4.6), (4.9) and $\sum_{i=1}^n v^i \neq 0$ any number v^i obeys the cubic equation $C_0 + C_1 v^i + C_2 (v^i)^2 + C_3 (v^i)^3 = 0$, with $C_3 \neq 0$, $i = 1, \dots, n$, and hence at most three numbers among v^i may be different. Thus, we are led to a contradiction. The proposition is proved.

This implies that in a future investigations of solutions to eqs. (4.8) and (4.9) for arbitrary n we will need a consideration of three non-trivial cases when 1) $v = (a, \dots, a)$ (see (4.11)); 2) $v = (a, \dots, a, b, \dots, b)$ ($a \neq b$); and 3) $v = (a, \dots, a, b, \dots, b, c, \dots, c)$ ($a \neq b$, $b \neq c$, $a \neq c$). One may put also $a > 0$ due to item **i**).

5 The generalization to the Lowelock model

The action (1.1) is a special case of the Lowelock model [34]

$$S = \int_M d^D z \sqrt{|g|} \left\{ \sum_{k=1}^m \alpha_k \mathcal{L}_k \right\}, \quad (5.1)$$

where $\alpha_1, \dots, \alpha_m$ are constants and \mathcal{L}_k are defined as follows

$$\mathcal{L}_k = 2^{-k} \delta_{N_1 \dots N_{2k}}^{M_1 \dots M_{2k}} R_{M_1 M_2}^{N_1 N_2} \dots R_{M_{2k-1} M_{2k}}^{N_{2k-1} N_{2k}}, \quad (5.2)$$

$k = 1, \dots, m$. (Usually, m is chosen as follows: $m = m(D) = [(D-1)/2]$; the terms with $k > m(D)$ will not give contributions into equations of motion.) Here

$$\delta_{N_1 \dots N_{2k}}^{M_1 \dots M_{2k}} = \sum_{\sigma} \varepsilon_{\sigma} \delta_{N_{\sigma(1)}}^{M_1} \dots \delta_{N_{\sigma(2k)}}^{M_{2k}} \quad (5.3)$$

is a generalized Kronecker tensor, totally antisymmetric in both groups of indices: M_1, \dots, M_{2k} and N_1, \dots, N_{2k} . In (5.3) a sum on all permutations of the set of indices $\{1, \dots, 2k\}$ is assumed. Here $\varepsilon_{\sigma} = \pm 1$ is the parity of the permutation σ .

It may be verified that $\mathcal{L}_1 = R[g]$ and \mathcal{L}_2 (from (5.2)) is coinciding with the Gauss-Bonnet term (1.2).

5.1 The Lagrange approach

Here we suggest the following conjecture: the equations of motion for the Lowelock action (5.1) when the metric (2.2) is substituted are equivalent to the Lagrange equations corresponding to the Lagrangian (for cosmological case see also [10, 28])

$$L = \sum_{k=1}^m \alpha_k L_k, \quad (5.4)$$

where

$$L_k = \mu_k \exp[-(2k-1)\gamma + \gamma_0] G_{i_1 \dots i_{2k}}^{(2k)} \dot{\beta}^{i_1} \dots \dot{\beta}^{i_{2k}}, \quad (5.5)$$

$\gamma_0 = \sum_{i=1}^n \beta^i$, μ_k are rational numbers ($\mu_1 = -w$, $\mu_2 = -1/3$) and

$$G_{i_1 \dots i_{2k}}^{(2k)} = \prod_{1 \leq r < s \leq 2k} (\delta_{i_r i_s} - 1) \quad (5.6)$$

are the components of Finslerian $2k$ -metric: $\langle v_1, \dots, v_{2k} \rangle_{2k} = G_{i_1 \dots i_{2k}}^{(2k)} v_1^{i_1} \dots v_{2k}^{i_{2k}}$, $v_s = (v_s^i) \in \mathbb{R}^n$, where $\langle \cdot, \dots, \cdot \rangle_{2k}$ is a $2k$ -linear symmetric form on \mathbb{R}^n , $k = 1, \dots, m$. Here $G_{i_1 i_2}^{(2)} = G_{i_1 i_2}$ and $G_{i_1 i_2 i_3 i_4}^{(4)} = G_{i_1 i_2 i_3 i_4}$, see (2.7) and (2.8).

5.2 Cosmological type solutions for “pure” m -th Lowelock model

Now we put $\alpha_1 = \dots = \alpha_{m-1} = 0$ and $\alpha_m \neq 0$, i.e. we consider the cosmological type model governed by the “pure” m -th Lowelock action

$$S_m = \alpha_m \int_M d^D z \sqrt{|g|} \mathcal{L}_m[g], \quad (5.7)$$

$m = 1, 2, 3, \dots$

It may be verified along a line as it was done in the Section 3 that our conjecture implies the existence of cosmological type solutions with the metrics (3.15) and (3.24).

For the “power-law” solution with the metric

$$g = w d\tau \otimes d\tau + \sum_{i=1}^n \varepsilon_i A_i^2 \tau^{2p^i} dy^i \otimes dy^i$$

the parameters p^i obey the following relations

$$\sum_{i=1}^n p^i = 2m - 1, \quad (5.8)$$

$$G_{i_1 \dots i_{2m}}^{(2m)} p^{i_1} \dots p^{i_{2m}} = (2m)! \sum_{i_1 < \dots < i_{2m}} p^{i_1} \dots p^{i_{2m}} = 0. \quad (5.9)$$

instead of (3.16) and (3.17). (For cosmological solutions see also [10, 28].)

For the “exponential” solution with the metric

$$g = wd\tau \otimes d\tau + \sum_{i=1}^n \varepsilon_i B_i^2 e^{2v^i \tau} dy^i \otimes dy^i$$

the parameters v^i should obey the relations (3.25): $\sum_{i=1}^n v^i = 0$ and

$$G_{i_1 \dots i_{2m}}^{(2m)} v^{i_1} \dots v^{i_{2m}} = (2m)! \sum_{i_1 < \dots < i_{2m}} v^{i_1} \dots v^{i_{2m}} = 0. \quad (5.10)$$

instead of (3.26).

The existence of these solutions corresponding to the “pure” Lowelock action (5.7) may be considered as test for the validity of the conjecture suggested above.

6 Conclusions

Here we have considered the $(n+1)$ -dimensional Einstein-Gauss-Bonnet model. For diagonal cosmological type metrics we have reduced the equation of motion to a set of Lagrange equations with the Lagrangian governed by two “minisuperspace” metrics on \mathbb{R}^n : (i) the pseudo-Euclidean 2-metric (corresponding to the scalar curvature term) and (ii) the Finslerian 4-metric (corresponding to the Gauss-Bonnet term). The Finslerian 4-metric is proportional to n -dimensional Berwald-Moor 4-metric. Thus, we have found a rather natural and “legitimate” application of n -dimensional Berwald-Moor metric in multidimensional gravity with the Gauss-Bonnet term. For the case of the “pure” Gauss-Bonnet model we have obtained two exact solutions: with power-law and exponential dependence of scale factors (w.r.t. “synchronous-like” variable). In the cosmological case (with $w = -1$, $\varepsilon_1 = \dots = \varepsilon_n = 1$) the first (power-law) solution was obtained earlier by N. Deruelle for $n = 4, 5$ [10] and verified by A. Toporensky and P. Tretyakov (for $n = 6, 7$) [13] and by S. Pavluchenko (for all n) [28]. See also [37].

When the “synchronous-like” time gauge was considered the equations of motion were reduced to an autonomous system of first order differential equations. It was shown that for any non-trivial solution with the exponential dependence of scale factors $a_i(\tau) = A_i \exp(v^i \tau)$, $i = 1, \dots, n$, there are no more than three different numbers among v^1, \dots, v^n (if $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$). This means that the solutions of such type have a “restricted” anisotropy. Such solutions may be used for constructing of new cosmological solutions, e.g. describing accelerated expansion of our 3-dimensional factor-space and small enough variation of the effective gravitational constant. For this approach, see [35, 36] and references therein.

We have also proposed (without a proof) a generalization of the EGB effective (cosmological-type) Lagrangian to the Lowelock case (in agreement with [10, 28] for cosmological metrics). According to this conjecture a “pure” Lowelock term of m -th order in the action gives a contribution to the effective Lagrangian that contains a Finslerian $2m$ -metric. This hypothesis implies the existence of cosmological solutions with power-law (see also [10, 28] for cosmological case) and exponential dependence of scale factors

for the case of the “pure” Lovelock model of m -th order. A proof of the conjecture mentioned above may be the subject of a separate publication. Another generalization of the approach suggested in this paper will be connected with inclusion of a scalar field.

Here an open problem arises: do the generalized solutions (for arbitrary n) with “jumping” parameters p^i, A_i appear as asymptotical solutions in EGB model when approaching a singular point? Recall that Kasner-type solutions with “jumping” parameters p^i, A_i describe an approaching to a singular point in certain gravitational models, e.g. with matter sources, see [38, 39, 40, 41, 42, 43, 44, 45, 46] and references therein. This problem may be a subject of separate investigations. (Here it is worth to mention the paper of T. Damour and H. Nicolai [47], which includes a study of the effect of the 4th order in curvature gravity terms, including the Euler-Lovelock term octic in velocities, and its compatibility with the Kac-Moody algebra E_{10} .)

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Appendix

A Useful relations for $(1 + n)$ -splitting

Let us consider the metric defined on $\mathbb{R}_* \times \mathbb{R}^n$ ($\mathbb{R}_* = (u_-, u_+)$ is an open subset in \mathbb{R})

$$g = we^{2\gamma(u)} du \otimes du + \sum_{i,j=1}^n h_{ij}(u) dy^i \otimes dy^j. \quad (\text{A.1})$$

Here $(h_{ij}(u))$ is a symmetric non-degenerate matrix for any $u \in \mathbb{R}_*$, smoothly dependent upon u . The function $\gamma(u)$ is smooth.

The calculations give the following non-zero (identically) components of the Riemann tensor

$$R_{0i0j} = -R_{i00j} = -R_{0ij0} = R_{i0j0} = \frac{1}{4}[-2\ddot{h}_{ij} + 2\dot{\gamma}\dot{h}_{ij} + \dot{h}_{ik}\dot{h}^{kl}\dot{h}_{lj}], \quad (\text{A.2})$$

$$R_{ijkl} = \frac{1}{4}(-w)e^{-2\gamma}(\dot{h}_{ik}\dot{h}_{jl} - \dot{h}_{il}\dot{h}_{jk}), \quad (\text{A.3})$$

$i, j, k, l = 1, \dots, n$, where here and in what follows $h^{-1} = (h^{ij})$ is the matrix inverse to the matrix $h = (h_{ij})$. Here we denote $\dot{A} = dA/du$ etc.

For non-zero (identically) components of the Ricci tensor we get

$$R_{00} = \frac{1}{2}[-h^{il}\ddot{h}_{li} + \frac{1}{2}h^{ij}\dot{h}_{jk}h^{kl}\dot{h}_{li} + h^{ik}\dot{h}_{ki}\dot{\gamma}], \quad (\text{A.4})$$

$$R_{ij} = \frac{1}{4}(-w)e^{-2\gamma}[2\ddot{h}_{ij} + \dot{h}_{ij}(h^{kl}\dot{h}_{lk} - 2\dot{\gamma}) - 2\dot{h}_{ik}h^{kl}\dot{h}_{lj}], \quad (\text{A.5})$$

$i, j = 1, \dots, n$.

The scalar curvature reads

$$R = \frac{1}{4}(-w)e^{-2\gamma}[4\text{tr}(\ddot{h}h^{-1}) + \text{tr}(\dot{h}h^{-1})(\text{tr}(\dot{h}h^{-1}) - 4\dot{\gamma}) - 3\text{tr}(\dot{h}h^{-1}\dot{h}h^{-1})]. \quad (\text{A.6})$$

Let us denote

$$M = \dot{h}h^{-1}, \quad (\text{A.7})$$

($h = (h_{ij})$), then

$$\dot{M} + M^2 = \ddot{h}h^{-1}. \quad (\text{A.8})$$

We obtain

$$R\sqrt{|g|} = L_1 + \frac{df_1}{du}, \quad (\text{A.9})$$

where

$$L_1 = \frac{1}{4}(-w)e^{-\gamma}\sqrt{|h|}[\text{tr}M^2 - (\text{tr}M)^2], \quad (\text{A.10})$$

$|h| = |\det(h_{ij})|$ and

$$f_1 = (-w)e^{-\gamma}\sqrt{|h|}\text{tr}M. \quad (\text{A.11})$$

In derivation of (A.9) the following relations were used:

$$\frac{d\sqrt{|h|}}{du} = \frac{1}{2}\sqrt{|h|}\text{tr}(\dot{h}h^{-1}), \quad \sqrt{|g|} = e^{\gamma}\sqrt{|h|}. \quad (\text{A.12})$$

The calculations give us the following relations for quadratic invariants

$$R_{MNPQ}R^{MNPQ} = \frac{1}{8}e^{-4\gamma}\{(\text{tr}M^2)^2 - \text{tr}M^4 + 2\text{tr}(2\dot{M} + M^2 - 2\dot{\gamma}M)^2\}, \quad (\text{A.13})$$

$$R_{MN}R^{MN} = \frac{1}{16}e^{-4\gamma}\{[-2\text{tr}\dot{M} - \text{tr}M^2 + 2\dot{\gamma}\text{tr}M]^2 + \text{tr}[2\dot{M} + (\text{tr}M - 2\dot{\gamma})M]^2\}. \quad (\text{A.14})$$

Relations (A.6), (A.13) and (A.14) imply the following formula for the Gauss-Bonnet term (1.2)

$$\begin{aligned} \mathcal{L}_2 = & \frac{1}{16}e^{-4\gamma}\{2(\text{tr}M^2)^2 - 2\text{tr}M^4 + [(\text{tr}M)^2 - \text{tr}M^2][8\text{tr}\dot{M} + 3\text{tr}M^2 \\ & + (\text{tr}M)^2 - 8\dot{\gamma}\text{tr}M] + 4\text{tr}[(M^2 - (\text{tr}M)M)(4\dot{M} - 4\dot{\gamma}M + M^2 + (\text{tr}M)M)]\}. \end{aligned} \quad (\text{A.15})$$

Relation (A.15) implies another important formula

$$\mathcal{L}_2\sqrt{|g|} = L_2 + \frac{d}{du}f_2, \quad (\text{A.16})$$

where

$$L_2 = \frac{1}{48}e^{-3\gamma}\sqrt{|h|}\{6\text{tr}M^4 - 3(\text{tr}M^2)^2 + 6\text{tr}M^2(\text{tr}M)^2 - 8(\text{tr}M)\text{tr}M^3 - \text{tr}M^4\} \quad (\text{A.17})$$

and

$$f_2 = \frac{1}{6}e^{-3\gamma}\sqrt{|h|}\{2\text{tr}M^3 - 3(\text{tr}M)\text{tr}M^2 + (\text{tr}M)^3\}. \quad (\text{A.18})$$

Diagonal metrics.

Now we consider the diagonal metric

$$h_{ij}(u) = e^{2\beta^i(u)}\varepsilon_i\delta_{ij}, \quad (\text{A.19})$$

$\varepsilon_i = \pm 1$, $i = 1, \dots, n$. Then, $M_{ij} = 2\dot{\beta}^i\delta_{ij}$ and we get the following relations for “Lagrangians”

$$L_1 = (-w)e^{-\gamma+\gamma_0} \left[\sum_{i=1}^n (\dot{\beta}^i)^2 - \left(\sum_{i=1}^n \dot{\beta}^i \right)^2 \right] \quad (\text{A.20})$$

$$L_2 = -\frac{1}{3}e^{-3\gamma+\gamma_0} \left\{ \left(\sum_{i=1}^n \dot{\beta}^i \right)^4 - 6 \left(\sum_{i=1}^n \dot{\beta}^i \right)^2 \sum_{j=1}^n (\dot{\beta}^j)^2 + 3 \left(\sum_{i=1}^n (\dot{\beta}^i)^2 \right)^2 + 8 \left(\sum_{i=1}^n \dot{\beta}^i \right) \sum_{j=1}^n (\dot{\beta}^j)^3 - 6 \sum_{i=1}^n (\dot{\beta}^i)^4 \right\}, \quad (\text{A.21})$$

where $\gamma_0 = \sum_{i=1}^n \beta^i$.

The “f-functions” (A.11) and (A.18) read as follows

$$f_1 = 2(-w)e^{-\gamma+\gamma_0} \sum_{i=1}^n \dot{\beta}^i, \quad (\text{A.22})$$

$$f_2 = \frac{4}{3}e^{-3\gamma+\gamma_0} \left[2 \sum_{i=1}^n (\dot{\beta}^i)^3 - 3 \left(\sum_{i=1}^n \dot{\beta}^i \right) \sum_{j=1}^n (\dot{\beta}^j)^2 + \left(\sum_{i=1}^n \dot{\beta}^i \right)^3 \right]. \quad (\text{A.23})$$

B Useful relations for Finslerian 4-metric

Here we consider a proof of identity (2.11). We decompose the product of 6 terms in the definition of the 4-metric (2.7) into the sum (of “powers of δ -s”)

$$G_{ijkl} = \sum_{a=0}^6 G_{ijkl}^a \quad (\text{B.1})$$

where

$$G_{ijkl}^0 = 1, \quad G_{ijkl}^1 = -\delta_{ij} - \delta_{ik} - \delta_{il} - \delta_{jk} - \delta_{jl} - \delta_{kl}, \dots, G_{ijkl}^6 = \delta_{ij}\delta_{ik}\delta_{il}\delta_{jk}\delta_{jl}\delta_{kl}.$$

Then we get

$$T = G_{ijkl}v^i v^j v^k v^l = \sum_{a=0}^6 T^a, \quad (\text{B.2})$$

where $T^a = G_{ijkl}^a v^i v^j v^k v^l$.

The calculations of T^a give us the following results:

$$\begin{aligned} T^0 &= S_1^4, & T^1 &= -6S_1^2 S_2, & T^2 &= 3S_2^2 + 12S_1 S_3, \\ T^3 &= -4S_1 S_3 - 16S_4, & T^4 &= 15S_4, & T^5 &= -6S_4, & T^6 &= S_4, \end{aligned} \quad (\text{B.3})$$

where

$$S_k = S_k(v) = \sum_{i=1}^n (v^i)^k, \quad (\text{B.4})$$

$k = 1, 2, 3, 4$.

The summation of all T^a in (B.3) leads us to the relation

$$T = G_{ijkl}v^i v^j v^k v^l = S_1^4 - 6S_1^2 S_2 + 3S_2^2 + 8S_1 S_3 - 6S_4 \quad (\text{B.5})$$

coinciding with (2.11).

Now we prove relation (4.6). We get

$$P_i = G_{ijkl}v^j v^k v^l = \sum_{a=0}^6 P_i^a, \quad (\text{B.6})$$

where $P_i^a = G_{ijkl}^a v^j v^k v^l$, $i = 1, \dots, n$.

The calculations of P_i^a give us the following formulas

$$\begin{aligned} P_i^0 &= S_1^3, & P_i^1 &= -3S_1^2 v^i - 3S_1 S_2, & P_i^2 &= 3S_3 + 3S_2 v^i + 9S_1 (v^i)^2, \\ P_i^3 &= -S_3 - 3S_1 (v^i)^2 - 16(v^i)^3, & P_i^4 &= 15(v^i)^3, & P_i^5 &= -6(v^i)^3, & P_i^6 &= (v^i)^3, \end{aligned} \quad (\text{B.7})$$

$i = 1, \dots, n$.

The summation of all P_i^a in (B.7) leads us to the relation

$$P_i = S_1^3 + 2S_3 - 3S_1 S_2 + 3(S_2 - S_1^2)v^i + 6S_1 (v^i)^2 - 6(v^i)^3, \quad (\text{B.8})$$

$i = 1, \dots, n$, coinciding with (4.6). This relation implies $P_i v^i = T$ in agreement with the definitions (B.2) and (B.6).

C Lagrange equations

Here we prove the relations (2.17)-(2.19) for the cosmological type metric (2.2) defined on manifold M from (2.1). The tensor \mathcal{E}_{MN} is obtained from the variation of the action

$$S = \int_M d^D z \sqrt{|g|} \mathcal{L}[g], \quad (\text{C.1})$$

with $\mathcal{L}[g] = \alpha_1 R[g] + \alpha_2 \mathcal{L}_2[g]$, i.e.

$$\delta S = \int_M d^D z \sqrt{|g|} \mathcal{E}_{MN} \delta g^{MN}, \quad (\text{C.2})$$

and $\sqrt{|g|} \mathcal{E}_{MN} = \delta S / \delta g^{MN}$.

Without loss of generality any 1-dimensional submanifold M_i is chosen to be compact and coinciding with the circle of unit length: $M_i = S_r^1$ ($r = 1/2\pi$ is the radius of the circle) and all coordinates y^i (see (2.1)) obey $0 < y^i < 1$, $i = 1, \dots, n$.

Here we will use the following relations for the components \mathcal{E}_{MN} in coordinates $(y^M) = (y^0 = u, y^i)$ and \mathcal{L} calculated for the metric (2.2):

$$\mathcal{E}_{MN} = \delta_{MN} \mathcal{E}_{NN}, \quad (\text{C.3})$$

$$\mathcal{E}_{MN} = \mathcal{E}_{MN}(u), \quad (\text{C.4})$$

$$\sqrt{|g|} \mathcal{L} = L + \frac{df}{du}, \quad (\text{C.5})$$

where $L = L(\gamma, \beta, \dot{\beta})$ and $f = f(\gamma, \beta, \dot{\beta})$ are defined in relations (2.4) and (2.9), respectively.

The first relation (C.3) may be readily verified using (2.14)-(2.16) and formulas for the Riemann tensor (A.2) and (A.3). The second relation (C.4) is an obvious one and the third one (C.5) is coinciding with (2.3).

The substitution of the metric (2.2) into the functional (C.1) gives us (due to (C.5) and $0 < y^i < 1$)

$$S = \int_{u_-}^{u_+} du \left(L + \frac{df}{du} \right) \quad (\text{C.6})$$

and hence

$$\delta S = \int_{u_-}^{u_+} du \left\{ \frac{\partial L}{\partial \gamma} \delta \gamma + \sum_{i=1}^n \left(\frac{\partial L}{\partial \beta^i} - \frac{d}{du} \frac{\partial L}{\partial \dot{\beta}^i} \right) \delta \beta^i \right\}, \quad (\text{C.7})$$

where $\delta \gamma(u)$ and $\delta \beta^i(u)$ are smooth functions with compact support in (u_-, u_+) ($\delta \gamma(u_{\pm}) = \delta \beta^i(u_{\pm}) = 0$), $i = 1, \dots, n$. On the other hand, using (C.2)-(C.4), the relation

$$(\delta g^{MN}) = \text{diag}(-2we^{-2\gamma} \delta \gamma, -2\varepsilon_1 e^{-2\beta^1} \delta \beta^1, \dots, -2\varepsilon_n e^{-2\beta^n} \delta \beta^n)$$

and $0 < y^i < 1$, we get

$$\delta S = \int_{u_-}^{u_+} du \{ \mathcal{E}_{00}(-2w) e^{\gamma_0 - \gamma} \delta \gamma + \sum_{i=1}^n \mathcal{E}_{ii}(-2\varepsilon_i) e^{\gamma + \gamma_0 - 2\beta^i} \delta \beta^i \}. \quad (\text{C.8})$$

Comparing (C.7) and (C.8) we get relations (2.17) and (2.18). Relations (2.19) just follow from (C.3).

D Riemann tensor squared

Here we consider the Riemann tensor squared (Kretschmann scalar) for the metric (3.15)

$$g = wd\tau \otimes d\tau + \sum_{i=1}^n \varepsilon_i A_i^2 \tau^{2p^i} dy^i \otimes dy^i.$$

From (A.13) we get

$$R_{MNPQ}R^{MNPQ} = K\tau^{-4}, \quad (\text{D.1})$$

where

$$K = 2S_4 + 2S_2^2 - 8S_3 + 4S_2 \quad (\text{D.2})$$

and $S_k = S_k(p) = \sum_{i=1}^n (p^i)^k$, $k = 1, 2, 3, 4$.

Using the identities

$$K = 4 \sum_{i=1}^n (p^i - 1)^2 (p^i)^2 + 2(S_2^2 - S_4). \quad (\text{D.3})$$

and

$$S_2^2 - S_4 = 2 \sum_{i < j} (p^i)^2 (p^j)^2 \quad (\text{D.4})$$

we obtain that $K \geq 0$ and $K = 0$ if and only if the set of parameters $p = (p^1, \dots, p^n)$ is either trivial: $p = (0, \dots, 0)$, or belongs to the Milne set:

$$p = (1, 0, \dots, 0), \dots, (0, \dots, 0, 1). \quad (\text{D.5})$$

For other sets p we have $K > 0$ and the Riemann tensor squared diverges when $\tau \rightarrow +0$.

E The proof of Proposition 1

The equations of motion (4.1) and (4.2) corresponding to the metric (3.15) with $h^i = p^i/\tau$ (here $\alpha_1 = 0$ and $\alpha_2 \neq 0$) read as follows

$$\mathcal{A} \equiv G_{ijkl} p^i p^j p^k p^l = 0, \quad (\text{E.1})$$

$$\mathcal{D}_i \equiv G_{ijkl} p^j p^k p^l = 0, \quad (\text{E.2})$$

$i = 1, \dots, n$.

Let $D = n + 1 \neq 4$ and

$$\mathcal{B} \equiv \frac{1}{(n-3)} \sum_{i=1}^n \mathcal{D}_i, \quad (\text{E.3})$$

$$\mathcal{C}_i \equiv \frac{1}{3} (\mathcal{B} - \mathcal{D}_i), \quad (\text{E.4})$$

$i = 1, \dots, n$.

For $D \neq 4$ the set of equations (E.1) and (E.2) is equivalent to the following set of equations

$$\mathcal{A} = S_1^4 - 6S_1^2 S_2 + 3S_2^2 + 8S_1 S_3 - 6S_4 = 24 \sum_{i < j < k < l} p^i p^j p^k p^l = 0, \quad (\text{E.5})$$

$$\mathcal{B} = (S_1 - 3)(S_1^3 - 3S_1 S_2 + 2S_3) = 6(S_1 - 3) \sum_{i < j < k} p^i p^j p^k = 0, \quad (\text{E.6})$$

$$\mathcal{C}_i = (S_1 - 3)p^i[2(p^i)^2 - 2S_1 p^i + S_1^2 - S_2] = 0, \quad (\text{E.7})$$

$i = 1, \dots, n$. Here $S_k = S_k(p) = \sum_{i=1}^n (p^i)^k$ and we used the identities (2.11), (4.6) and the following identity

$$S_1^3 - 3S_1 S_2 + 2S_3 = G_{ijk} p^i p^j p^k = 6 \sum_{i < j < k} p^i p^j p^k, \quad (\text{E.8})$$

where

$$G_{ijk} = (\delta_{ij} - 1)(\delta_{ik} - 1)(\delta_{jk} - 1) \quad (\text{E.9})$$

are components of a Finslerian 3-metric. The identity (E.8) could be readily verified along a line as it was done in Appendix B for the Finslerian 4-metric. (We note that relation (E.6) may be also obtained using the formula (A.16).)

For $S_1 = 3$ we obtain the main solution governed by relations (3.16) and (3.17).

Now we consider another case $S_1 \neq 3$. Let k be the number of all nonzero numbers among p^1, \dots, p^n . For $k = 0$ we get a trivial solution $(0, \dots, 0)$. Let $k \geq 1$. We suppose without loss of generality that p^1, \dots, p^k are nonzero. For $k = 1, 2$ all relations (E.5)-(E.7) are satisfied identically. In all three cases $k = 0, 1, 2$ the solutions have the form $(a, b, 0, \dots, 0)$ (plus permutations for general setup).

Now we consider $k \geq 3$. From (E.7) and $S_1 \neq 3$ we obtain

$$2(p^i)^2 - 2S_1 p^i + S_1^2 - S_2 = 0, \quad (\text{E.10})$$

$i = 1, \dots, k$. Summing on i gives us $(2 - k)(S_2 - S_1^2) = 0$, or $S_2 = S_1^2$. Then we obtain from (E.6) $S_3 = S_1^3$ and from (E.5): $S_4 = S_1^4$. Thus, we get $S_4 = S_2^2$ implying $\Sigma = \sum_{1 \leq i < j \leq k} (p^i)^2 (p^j)^2 = 0$. But $\Sigma \geq (p^1)^2 (p^2)^2 > 0$. Hence, we are led to a contradiction. That means that for $S_1 \neq 3$, we have only solutions with $k \leq 2$ of the form $(a, b, 0, \dots, 0)$ (plus permutations for general setup). The Proposition 1 is proved.

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